

MAXIMAL COHEN-MACAULAY MODULES OVER LOCAL TORIC RINGS

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ABSTRACT. In analogy with the classical, affine toric rings, we define a local toric ring as the quotient of a regular local ring modulo an ideal generated by binomials in a regular system of parameters with unit coefficients; if the coefficients are just ± 1 , we call the ring purely toric. We prove the following results on the existence of maximal Cohen-Macaulay modules: (EQUI _{p}) we construct certain families of local toric rings satisfying Hochster's small Cohen-Macaulay conjecture in positive characteristic; (EQUI₀) provided Hochster's small Cohen-Macaulay conjecture holds in positive characteristic with the additional condition that the multiplicity of the small Cohen-Macaulay module is bounded in terms of the parameter degree of the ring, then any local ring (not necessarily toric) in equal characteristic zero admits a formally étale extension satisfying Hochster's small Cohen-Macaulay conjecture (this applies in particular to the families from (EQUI _{p})); and (MIX), in mixed characteristic, we show that all purely toric local rings satisfy Hochster's big Cohen-Macaulay conjecture, and so do those belonging to the families from (EQUI _{p}).

1. INTRODUCTION

Recall that a (not necessarily finitely generated, aka *big*) module Q over a Noetherian local ring (R, \mathfrak{m}) is called *Cohen-Macaulay*, if $\dim M = \text{depth}(M)$. One is especially interested in the case where this common value is also the dimension of R , and we will refer to such a module as a *maximal Cohen-Macaulay* module, and abbreviate it as MCM.¹ When an MCM is finitely generated, we emphasize this by calling it *small*. By taking completion, one can always ensure that Q satisfies the following stronger property: any system of parameters on R is Q -regular (Q is then a so-called *balanced MCM*), and we will henceforth just mean this latter, when discussing the existence of MCM's.

The interest of these notions, introduced by Hochster, is that a Noetherian local ring admitting an MCM satisfies almost all of the homological conjectures; for Serre's positivity conjecture one needs the MCM to be moreover small (see §3 for more details). Since all homological conjectures admit faithfully flat descent, there is no loss of generality in proving the existence of (big or small) MCM's after taking a scalar extension (in the sense of [19, §3]), and so we may assume that R is furthermore complete and has algebraically closed residue field, which we may even take to have uncountable cardinality. Moreover, in the mixed characteristic case, to avoid pathologies, we require that R is torsion-free over some complete p -ring (=mixed characteristic complete discrete valuation ring with uniformizing parameter p). To not have to repeat all these conditions in this introduction, we

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¹Hochster and others sometimes leave out the 'maximal', which might lead to confusion.

call such a ring temporarily *analytic*. The current state of the existence of MCM's for analytic rings, leaving aside some special cases, can be summed up as follows:

Theorem ([9, 10, 11]). *A d -dimensional analytic ring admits*

Hochster-Huneke: *in equal characteristic, a big MCM algebra;*

Hochster/Heitmann: *in mixed characteristic and $d \leq 3$, a big MCM;*

Hochster/Hartshorne/Peskine-Szpiro: *if $d = 2$, or, in the graded case, if $d = 3$ and the characteristic is positive, a small MCM.*

So the two major open problems are: (H_{big}) the existence of big MCM's in mixed characteristic in dimensions four and higher; and (H_{small}) the existence of small MCM's, in any characteristic, in dimension three and higher.² The most pressing of these is (H_{big}) , as many homological conjectures are still outstanding in mixed characteristic. On the other hand, Hochster seems skeptical about (H_{small}) if R does not have positive characteristic. We will formulate a slightly stronger version of (H_{small}) and dissuade Hochster's distrust in equal characteristic zero. Namely, let us say that an MCM is *very small* if its multiplicity is at most the equi-parameter degree of R . Recall that the *parameter degree* $\text{pardeg}(R)$ of R is the smallest possible length of R/I where I varies over all parameter ideals (=ideals generated by a system of parameters); in mixed characteristic p , we must use instead the *equi-parameter degree* $\text{epardeg}(R)$, which is the smallest possible length of R/I where I varies over all parameter ideals containing p . Since the parameter degree of R is equal to its multiplicity if and only if R is Cohen-Macaulay, local Cohen-Macaulay rings trivially admit a very small MCM, and I now conjecture more generally:

1.1. Conjecture. *Any complete local ring admits a very small MCM.*

It is the latter form of (H_{small}) that can be lifted to equal characteristic zero, by an ultraproduct argument (which requires the residue field to be uncountable; see §7):

1.2. Theorem. *If every d -dimensional analytic ring of positive characteristic admits a very small MCM, then so does any d -dimensional analytic ring of equal characteristic.*

So, we should focus on problem (H_{small}) for analytic rings of positive characteristic, under the additional very smallness condition, and we will now describe a class of analytic rings for which we can show the validity of the conjecture.

Recall that an *affine toric variety* over a field κ is an irreducible variety containing a torus $(\kappa^*)^n$ as a Zariski open subset such that the action of the torus on itself extends to an algebraic action on the whole variety. Hochster [7] proved that a normal toric variety is automatically Cohen-Macaulay. Since the integral closure of a toric variety is again toric, it follows that any local ring of a toric variety admits a small MCM. To generalize this, we use the following algebraic characterization of toric varieties (see, for example, [3, §1.1]): the ideals defining affine toric varieties are precisely the prime ideals generated by pure binomials (=polynomials in the variables $\mathbf{y} := (y_1, \dots, y_n)$ of the form $\mathbf{y}^\alpha \pm \mathbf{y}^\beta$ with $\alpha, \beta \in \mathbb{N}^n$).³ In [4], this was

²The one known three-dimensional (graded, positive characteristic) is really the case of a cone over a two-dimensional variety; likewise, the few known higher-dimensional cases are somehow derived from lower dimensional cases, like [5].

³Following common practice, we write \mathbf{y}^α for $y_1^{a_1} \cdots y_n^{a_n}$, for $\alpha = (a_1, \dots, a_n)$.

taken as the departing point of investigating general *toric* or *binomial* ideals in a polynomial ring over a field as those ideals generated by binomials (with arbitrary non-zero coefficients, not just ± 1). We consider the following local version:

1.3. Definition. Given an unramified⁴ regular local ring S and a regular system of parameters \mathbf{y} in S , call an ideal $I \subseteq S$ *toric* (with respect to the given regular system of parameters), if it is generated by ‘binomials’ of the form $\mathbf{y}^\alpha - u\mathbf{y}^\beta$, where $u \in S^*$ is a unit in S . More generally, if $\Xi \subseteq S^*$ is a subgroup, then we call the ideal Ξ -*toric*, if all coefficients u belong to Ξ . In particular, if $\Xi = \{\pm 1\}$, we call the ideal *purely toric*. The corresponding quotient S/I will be called a (Ξ) -*toric local ring*.

Since we work locally, there are many more units, and so we get a much larger class than in the affine case (e.g., not only the cusp $0 = x^2 - y^3$ but also the node $0 = x^2 - y^2 - y^3 = x^2 - (1 + y)y^2$ is toric at the origin). We are mainly interested in analytic toric rings, that is to say, when S is a power series ring, either over an uncountable algebraically closed field or over a complete p -ring, and so we would like to prove that any toric analytic ring admits a small MCM.

Mixed characteristic. We give the following positive solution of (H_{big}) :

1.4. Theorem. *Every purely toric analytic ring in mixed characteristic admits a big MCM.*

We prove in fact a more general theorem (Theorem 7.4): using Witt vectors, we define the *Witt transform* $R^{\mathcal{W}}$ of an analytic ring R ; it is again analytic, of the same dimension, and we call R *Witt-closed*, if $R \cong R^{\mathcal{W}}$. We then show that any Witt-closed local ring admits a big MCM. The theorem follows since the multiplicativity of the Teichmüller character implies that purely toric rings are Witt-closed. There is a more general class of toric rings for which we can prove Theorem 1.4, which we will discuss below. It should be pointed out though that our construction is different from the one in [8].

Equal characteristic. Fix an uncountable algebraically closed field κ of characteristic $p \geq 0$. The κ^* -toric case can be reduced to the affine case, using Hochster’s result (where κ^* is the multiplicative group of κ). However, we will show that for a toric domain R with $p > 0$, even in the affine case, there is a candidate for an MCM which is usually smaller than its normalization—in particular, it is very small—, namely the ring $\mathbf{F}^{\text{int}}(R)$ of *F-integral elements* (=the elements in its field of fraction such that some p^n -th power lies in R); see Conjecture 6.4 for a characteristic-free variant. To have such ‘smaller’ MCM’s might be advantageous for calculating intersection forms, as we will briefly explain in §3. Although we cannot prove the result in general, we will prove it for certain families of toric analytic rings. More concretely, consider the following families $\mathcal{T}_{d,n,m}(V)$ of analytic rings: let V be either a field or a complete p -ring, and let $\mathcal{T}_{d,n,m}(V)$ consist of all $R := S[[\mathbf{u}]]/I$, where $S := V[[y_1, \dots, y_d]]$ and $\mathbf{u} = (u_1, \dots, u_n)$, where \mathbf{y} is either (y_1, \dots, y_d) (equal characteristic case) or (p, y_1, \dots, y_d) (mixed characteristic case), and where I is a toric ideal generated by

⁴Complete ramified regular local rings might not be power series rings, a technical complication I prefer to avoid, although I do not know how essential it is.

- ($n \geq 2, m = 2$): binomials of the form $u_i^2 - a_i \mathbf{y}^{\alpha_i}$ and $u_i u_j - b_{ij} \mathbf{y}^{\beta_{ij}}$, with a_i, b_{ij} units in S such that $a_i a_j = b_{ij}^2$ and $\alpha_i + \alpha_j = 2\beta_{ij}$;
 ($n = 2, m$ **odd**): binomials of the form $u_i^m - a_i \mathbf{y}^{\alpha_i}$, for $i = 1, 2$, and $u_1 u_2 - b \mathbf{y}^\beta$, a_1, a_2, b units in S such that $a_1 a_2 = b^m$ and $\alpha_1 + \alpha_2 = m\beta$.

Note that the conditions on the a_i and α_i guarantee that in either case the ideal I has height n (whence $\dim R = d + \dim V$) and $S \subseteq R$ is finite, yielding a ‘toric’ Noether normalization. We prove the following:

1.5. Theorem. *For R an analytic ring belonging to $\mathcal{T}_{d,n,m}(V)$,*

- (EQUI) *if V is an uncountable algebraically closed field whose characteristic does not divide m , then R admits a very small MCM;*
 (MIX) *if V is a complete p -ring, then R admits a big MCM.*⁵

For the remainder of the introduction, I will now explain the first case, when $V = \kappa$ is an uncountable algebraically closed field, and by a variant of Theorem 1.2 (see Remark 6.1), we only need to consider the case that the characteristic p is positive. The rings R in $\mathcal{T}_{d,n,m}(\kappa)$ are not domains, but they all admit a d -dimensional, minimal prime ideal \mathfrak{p} which is toric. We will show that the F-integral elements of R/\mathfrak{p} form a small MCM over R/\mathfrak{p} whence over R . (For a domain A of characteristic p , we call an element f in its field of fractions *F-integral*, if $f^q \in A$ for some $q = p^n$.) However, it is useful to have a construction that does not require the ring to be a domain, and for that, we have to look at the *Frobenius transform* of R : viewing R as a module over itself via the Frobenius $\mathbf{F}_p: R \rightarrow R$, we will denote it as $\mathbf{F}_* R$; likewise, elements of $\mathbf{F}_* R$ will be denoted by $*r$, so that the scalar action is given by $s*r = *s^p r$. Our desired MCM will be an R -submodule Q of $\mathbf{F}_* R$ (where for small p we may have to take instead some iterate of \mathbf{F}_p). Concretely, Q is the S -saturation of the (cyclic) R -submodule generated by $*1$: namely, the S -submodule of $\mathbf{F}_* R$ consisting of all $*r$ such that $s*r = a*1$ for some non-zero $s \in S$ and $a \in R$. It follows that Q is again an R -module, but the surprising fact is that it is free as an S -module (of rank m), whence a very small MCM (the proof of Theorem 1.2 then automatically also proves the equal characteristic zero case). However, this freeness result is still an enigma: I can write down explicitly m generators over S (which are easily seen to be linearly independent over S), but I do not know how to find these generators for more complicated toric ideals. At the moment, everything relies on some basic congruence (arithmetic) relations, but it is not clear what these would become in a more general setting.

The toric rings in the families \mathcal{T} are in fact special cases of a more general construction of toric rings: recall that in the affine case, a toric ideal can also be defined in terms of a (partial) character defined on a sublattice $\Gamma \subseteq \mathbb{Z}^n$ ([4, Theorem 2.1]). We will consider below the case where Γ is moreover the graph of a morphism (the resulting binomial equations are then ‘bi-partite’, that is to say, of the form $\mathbf{x}^\alpha - u\mathbf{y}^\beta$, with (\mathbf{x}, \mathbf{y}) a fixed partition of the variables). Although I cannot show (EQUI) in Theorem 1.5 for this larger class of rings, I give an example where we verify the conjecture. On the other hand, we do verify (MIX) in general for this class.

A note of caution: we cannot expect that for non-toric analytic domains, their ring of F-integral elements is always MCM: Bhatt [2] has given examples in positive

⁵Although these are classes of non-purely toric ideals, in mixed characteristic, Theorem 1.4 also applies to them by a specialization argument, Theorem 7.7.

characteristic⁶ of normal analytic domains R that do not admit a small MCM which in addition carries the structure of an R -algebra. In a future paper, we will give a new condition (involving local cohomology) for the existence of a small MCM and conjecture that this condition can be realized inside the Frobenius transform \mathbf{F}_*M of some, possibly non-free, R -module M .

2. HOCHSTER'S MCM CONJECTURES AND HOCHSTER RINGS

Recall that a *scalar extension* of local rings $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a formally etale homomorphism, that is to say, faithfully flat and unramified (meaning that $\mathfrak{m}S = \mathfrak{n}$). In particular, R and S have the same dimension and depth. Moreover, if κ is the residue field of R and λ any field extension of κ , then there exists a unique scalar extension R_λ^\wedge of R which is complete and has residue field λ (see, for instance, [19, §3]). The philosophy for allowing scalar extensions is that they do not change the singularities and that any property that descends under faithfully flat maps is inherited from the scalar extension.

Let us therefore for call a local ring R *Hochster*, if some scalar extension of R admits a small MCM, that is to say, a finitely generated module of depth equal to the dimension of R . It is well-known that any two-dimensional local ring is Hochster. By work of Hartshorne, Peskine-Szpiro, and Hochster, any three-dimensional graded domain in positive characteristic is Hochster (see, for instance [11, Corollary 2]). We will call R *strongly Hochster*, if some scalar extension of R admits a very small MCM. Given two local rings S and R , let us say that R is a *quasi-deformation* of S , if there is a diagram

$$(1) \quad \begin{array}{ccc} & & S \\ & & \downarrow \\ R & \xrightarrow{g} & \bar{S} \end{array}$$

with g finite and injective and $\bar{S} = S/(x_1, \dots, x_n)S$ with (x_1, \dots, x_n) part of a system of parameters in S . We sometimes refer to the whole diagram (1) as a quasi-deformation. The completion of a quasi-deformation is again a quasi-deformation, as is any flat base change.

2.1. Lemma. *Let R be a quasi-deformation of S . If S admits a big MCM, then so does R . If S is moreover Hochster, then so is R .*

Proof. In either case, we may replace R by its completion and assume from the start that we have a quasi-deformation (1) with R and S complete. Let κ and λ be the respective residue fields of R and S . Let Q be an MCM module over S , which we may assume to be balanced, and put $\bar{Q} := Q/(x_1, \dots, x_n)Q$. Let (y_1, \dots, y_d) be a system of parameters of R , which therefore is also a system of parameters in \bar{S} . Let us continue to write y_i for a choice of lifting in S . Hence $(x_1, \dots, x_n, y_1, \dots, y_d)$ is a system of parameters in S , whence is Q -regular. It follows that (y_1, \dots, y_d) is \bar{Q} -regular, showing that \bar{Q} is an MCM over R . If S is moreover Hochster, then there exists a scalar extension $S \rightarrow S'$ such that Q is a finite S' -module. We may replace S' and Q by their completion, and hence $S' \cong S_\lambda^\wedge$ by [19, Proposition 3.4],

⁶In equal characteristic, normal domains split off in any finite extension, and so does their local cohomology, so no finite extension of a non-Cohen-Macaulay normal domain can be Cohen-Macaulay.

where λ' is the residue field of S' . Therefore, $R' := R_{\lambda'}^{\wedge}$ is a scalar extension of R and \bar{Q} is finitely generated as an R' -module. \square

Hence, in view of Conjecture 4.2, below, we may ask which complete local rings are quasi-deformations of toric local rings. Bhatt's counterexamples to the existence of small MCM algebras ([2]) shows that, if we believe in the conjecture, then not every local ring can be obtained this way.

3. CALCULATING INTERSECTION FORMS USING SMALL MCM'S

As mentioned above, the existence of an MCM has major implications for the homological theory of a ring. In particular, many homological conjectures remain open in mixed characteristic since we do not know yet the existence of MCM's in that case. For most applications, it suffices to find a big MCM, but there is at least one case where this is not enough: to prove the positivity conjecture of Serre for ramified regular local rings in mixed characteristic (the remaining open case). To this end, it suffices by faithfully flat descent to show that local rings in mixed characteristic are Hochster. But even in equal characteristic it is useful to show that local rings are Hochster: although the positivity conjecture is known in that case, small MCM's can aid in calculating the intersection form, as I will now briefly explain.

Let X be a variety, $Y, Z \subseteq X$ closed subvarieties, and x a point in their intersection $Y \cap Z$. Suppose x is a regular point on X , and the intersection at x is *in general position*, in the sense that the sum of the local dimensions of Y and Z at x is at most the local dimension of X at x and $Y \cap Z$ is zero-dimensional at x . The latter means that $\mathcal{O}_{Y,x} \otimes \mathcal{O}_{Z,x}$ has finite length, and this finite length $\ell(\mathcal{O}_{Y,x} \otimes \mathcal{O}_{Z,x})$ is called the '*naive*' intersection multiplicity of Y and Z at x . If Y and Z are curves in the plane, this leads to the correct Bezout formula, meaning that the sum over all intersection points is the product of the degrees of the curves. However, Serre realized that this is no longer true in higher dimensions due to the lack of Cohen-Macaulayness, and he suggested the following *local intersection multiplicity*:

$$(2) \quad \chi(Y, Z; x) := \sum_{i=0}^{\infty} (-1)^i \ell(\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y,x}, \mathcal{O}_{Z,x}))$$

Note that the sum is well-defined since all higher Tor's also must have finite length and vanish for i bigger than the dimension of X . We need a definition that includes also modules, and for this it is easier to formulate everything in terms of local rings. Hence, let (S, \mathfrak{n}) be a d -dimensional regular local, let M and N be S -modules such that $M \otimes N$ has finite length, meaning that $\mathrm{Ann}(M) + \mathrm{Ann}(N)$ is \mathfrak{n} -primary, and $\dim M + \dim N \leq d$. We then define their intersection form similarly as

$$(3) \quad \chi(M, N) := \sum_{i=0}^d \mathrm{Tor}_i^S(M, N).$$

Serre proved that $\chi(M, N) \geq 0$, with equality if and only if $\dim M + \dim N < d$, whenever S is equicharacteristic or unramified. To discuss the use of small MCM's, we need a vanishing theorem.

3.1. Lemma. *Let R be a d -dimensional local ring and let M and N be R -modules such that $M \otimes N$ has finite length. Then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > \mathrm{pd}(M) + \mathrm{pd}(N) - d$.*

Proof. There is nothing to prove if either module has infinite projective dimension, so assume the modules have moreover finite projective dimension. Let i be maximal such that there exists a pair of R -modules M and N of finite projective dimension p and q respectively, such that $M \otimes N$ has finite length and $\mathrm{Tor}_i^R(M, N) \neq 0$. We need to show that $i \leq p + q - d$, so suppose not, that is to say, $i > p + q - d$. Note that for $i > q$ we automatically have $\mathrm{Tor}_i^R(M, N) = 0$, showing that necessarily $p < d$. By Auslander-Buchsbaum therefore, M must have positive depth (and likewise N). Let x be an M -regular element. Since $M \otimes N$ has finite length, some power x^n lies in $\mathrm{Ann}(M) + \mathrm{Ann}(N)$, say, $x^n = a + b$ with $aM = 0$ and $bN = 0$. Hence replacing x by b , we may moreover assume that $xN = 0$. We have an exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$, which tensoring with N gives part of a long exact sequence

$$(4) \quad \mathrm{Tor}_{i+1}^R(M/xM, N) \rightarrow \mathrm{Tor}_i^R(M, N) \xrightarrow{-x} \mathrm{Tor}_i^R(M, N).$$

By Auslander-Buchsbaum, $\mathrm{pd}(M/xM) = p + 1$, and so $i + 1 > p + 1 + q - d$. Since $(M/xM) \otimes N$ too has finite length, maximality of i yields $\mathrm{Tor}_{i+1}^R(M/xM, N) = 0$. On the other hand, x is zero on N whence on $\mathrm{Tor}_i^R(M, N)$, so that (4) implies that also $\mathrm{Tor}_i^R(M, N) = 0$, contradiction. \square

MCM's come in via the following corollary, which implies that the 'naive intersection form' is the correct one in the Cohen-Macaulay case:

3.2. Corollary. *Let S be a regular local ring of dimension d , let M and N be Cohen-Macaulay S -modules such that $M \otimes N$ has finite length and $\dim M + \dim N = d$. Then $\chi(M, N) = \ell(M \otimes N) > 0$.*

Proof. Let p and q be the respective dimensions, whence depths of M and N , so that $d = p + q$ by assumption. By Auslander-Buchsbaum, $\mathrm{pd}(M) = d - p$ and $\mathrm{pd}(N) = d - q$, and hence $\mathrm{Tor}_i^R(M, N) = 0$ by Lemma 3.1, for all $i > d - p + d - q - d = 0$. \square

We can now describe an algorithm for calculating this intersection form, in case we have explicit, small MCM's: let \mathfrak{p} and \mathfrak{q} be prime ideals in S whose sum is \mathfrak{n} -primary, with respective dimensions p and $q = d - p$. Let M be a small MCM over S/\mathfrak{p} , and similarly N a small MCM over S/\mathfrak{q} . Consider a prime filtration $M_s \subseteq \cdots \subseteq M_2 \subseteq M$ of M , meaning that each subsequent quotient M_i/M_{i+1} is of the form S/\mathfrak{g}_i with each \mathfrak{g}_i some prime ideal in the support of M . It is well-known that \mathfrak{p} appears exactly $a := \ell(M_{\mathfrak{p}})$ times among the \mathfrak{g}_i . Since $\chi(\cdot, N)$ is additive, and since it vanishes on modules of dimension less than $d - q = p$, whence in particular on the S/\mathfrak{g}_i with $\mathfrak{g}_i \neq \mathfrak{p}$, we get $\chi(M, N) = \sum \chi(S/\mathfrak{g}_i, N) = a\chi(S/\mathfrak{p}, N)$. Similarly, with $b := \ell(N_{\mathfrak{q}})$, the same argument then yields that

$$\chi(M, N) = a\chi(S/\mathfrak{p}, N) = ab\chi(S/\mathfrak{p}, S/\mathfrak{q})$$

By Corollary 3.2, however, we know that $\chi(M, N) = \ell(M \otimes N)$, so that we obtained the following formula for the intersection form of two varieties in terms of the naive intersection form of their small MCM's:

$$(5) \quad \chi(S/\mathfrak{p}, S/\mathfrak{q}) = \frac{\ell(M \otimes N)}{\ell(M_{\mathfrak{p}})\ell(N_{\mathfrak{q}})}.$$

In particular, if one of the varieties is already Cohen-Macaulay, say S/\mathfrak{p} , so that we may take $M = S/\mathfrak{p}$, the formula becomes

$$(6) \quad \chi(S/\mathfrak{p}, S/\mathfrak{q}) = \frac{\ell(N/\mathfrak{p}N)}{\ell(N_{\mathfrak{q}})}.$$

Suppose N is very small, so that $\ell(N_{\mathfrak{q}}) \leq \text{mult}(N) \leq \text{epardeg}(S/\mathfrak{q})$ by [21, Theorem 6.4] and since N is generated by at most $\ell(N/\mathfrak{p}N)$ elements, we get a lower bound

$$\frac{\mu(N)}{\text{epardeg}(S/\mathfrak{q})} \leq \chi(S/\mathfrak{p}, S/\mathfrak{q}).$$

4. THE RING OF F-INTEGRAL ELEMENTS

Let R be a Noetherian domain of characteristic $p > 0$ and K its field of fractions.

4.1. Definition. Let q be a power of the characteristic p . We say that an element $f \in K$ is q -integral if $f^q \in R$. One easily checks that the sum and product of q -integral elements is again q -integral, so that they form a subring $\mathbf{F}_q^{\text{int}}(R)$ of K , called the *ring of q -integral elements*.

Clearly, $\mathbf{F}_q^{\text{int}}(R)$ lies inside the integral closure \tilde{R} of R . If $R = \mathbf{F}_p^{\text{int}}(R)$, then we say that R is F -normal. We always have a chain of subrings $R \subseteq \mathbf{F}_p^{\text{int}}(R) \subseteq \mathbf{F}_{p^2}^{\text{int}}(R) \subseteq \cdots \subseteq \tilde{R}$, which eventually must stabilize since \tilde{R} is finite over R . This stable value, say $\mathbf{F}_q^{\text{int}}(R)$, is therefore an F -normal ring, called the *F -normalization* of R and will simply be denoted $\mathbf{F}^{\text{int}}(R)$. Elements in $\mathbf{F}^{\text{int}}(R)$ will simply be called *F -integral*. It is not hard to see that R is F -normal if and only if every principal ideal is Frobenius closed (compare this with the fact that R is normal if and only if every principal ideal is tightly closed; see, for instance, [12] for details on these closure operations).

We cannot expect $\mathbf{F}^{\text{int}}(R)$ to be always a small MCM for R : indeed, if R is normal but not Cohen-Macaulay, then $R \subseteq \mathbf{F}_q^{\text{int}}(R) \subseteq \tilde{R}$ are all equal, but not Cohen-Macaulay.⁷ For affine toric domains, however, normal implies Cohen-Macaulay, by a result of Hochster, and one could expect the same to hold for local toric domains. We can now conjecture the following sharper version in positive characteristic:

4.2. Conjecture. *If R is a complete toric domain of positive characteristic, then $\mathbf{F}^{\text{int}}(R)$ is Cohen-Macaulay, whence in particular a small MCM algebra.*

We now give an alternative description of $\mathbf{F}^{\text{int}}(R)$, which will allow us to verify the conjecture in certain cases. Moreover, the construction also works for non-domains. In fact, we are not interested in the ring structure on $\mathbf{F}^{\text{int}}(R)$, but instead will realize it as a certain submodule of \mathbf{F}_*R . Recall that $\mathbf{F}_q: R \rightarrow R$ denotes the Frobenius morphism $x \mapsto x^q$, where q is some power of the characteristic p . Viewing R as an R -module via this morphism, we get an R -module which we denote by $\mathbf{F}_{q*}R$, or just \mathbf{F}_*R , if q is clear from the context; we call it the *Frobenius transform* of R (with respect to q). We will denote elements in \mathbf{F}_{q*} by $*_qa$, or just $*a$ if q is clear from the context. The R -module action on $\mathbf{F}_{q*}R$ is now conveniently given by $r*a := *r^qa$. We make \mathbf{F}_*R into an R -algebra, by giving it a ring structure via

⁷As explained in the introduction, a more serious obstruction is given by the counterexamples in [2].

$(*b) \cdot (*c) := *bc$. Hence as abstract rings, R and \mathbf{F}_*R are isomorphic, but not as R -algebras. In fact, by Kunz's theorem, \mathbf{F}_*R is flat as an R -module, of rank q^d , if and only if R is a d -dimensional regular ring.

Suppose R is a domain with field of fractions K . We may likewise view $\mathbf{F}_{q*}K$ as a K -algebra, whence by restriction as an $\mathbf{F}_q^{\text{int}}(R)$ -algebra. One easily checks that $\mathbf{F}_{q*}R$ is invariant under $\mathbf{F}_q^{\text{int}}(R)$, so that $\mathbf{F}_{q*}R$ has the structure of an $\mathbf{F}_q^{\text{int}}(R)$ -algebra. In fact, we will shortly realize $\mathbf{F}_q^{\text{int}}(R)$ as an R -submodule of $\mathbf{F}_{q*}R$.

Let R be a Noetherian ring, $\Sigma \subseteq R$ a multiplicative set in R , let M be an R -module and $N \subseteq M$ a submodule. We say that N is Σ -saturated in M , if $sm \in N$ for some $m \in M$ and $s \in \Sigma$ implies that already $m \in N$. This is equivalent with the canonical map $M \rightarrow \Sigma^{-1}(M/N)$ being injective. We define the Σ -saturation $\text{sat}_\Sigma(N; M)$ of N in M to be the submodule of all $m \in M$ such that $sm \in N$ for some $s \in \Sigma$. It is not hard to see that $\text{sat}_\Sigma(N; M)$ is the kernel of $M \rightarrow \Sigma^{-1}(M/N)$. In particular, it is again an R -module. In case N is cyclic, generated by a single element n , we may write $\text{sat}_\Sigma(n; M)$ instead of $\text{sat}_\Sigma(N; M)$.

We will use this construction in the following situation. Let R be a complete Noetherian local ring of characteristic $p > 0$ and let $S \subseteq R$ be some Noether normalization of R , that is to say, a regular local subring $S \subseteq R$, over which R is finite (as an S -module). Recall that by Cohen's structure theorem, Noether normalizations which have moreover the same residue field, are completely determined by a choice of system of parameters in R , by having them act as the variables in a power series over the residue field. Note that $S \setminus \{0\}$ is a multiplicative set in R , which, for simplicity, we just denote by S .

4.3. Proposition. *If R is a complete local domain of characteristic $p > 0$, if q is any power of p , and if $S \subseteq R$ is some Noether normalization of R , then $\text{sat}_S(*1; \mathbf{F}_{q*}R)$ is the $\mathbf{F}_q^{\text{int}}(R)$ -submodule of $\mathbf{F}_{q*}R$ generated by $*1$, and hence $\text{sat}_S(*1; \mathbf{F}_{q*}R) \cong \mathbf{F}_q^{\text{int}}(R)$, so that in particular it is independent from the Noether normalization S .*

Proof. An element in the $\mathbf{F}_q^{\text{int}}(R)$ -submodule of $\mathbf{F}_{q*}R$ generated by $*1$ is of the form $f*1$ for some $f \in \mathbf{F}_q^{\text{int}}(R)$. The latter means that $f^q = a \in R$, and so $f*1 = *a$. Write $f = r/t$ with $r, t \in R$ and $t \neq 0$. Since R is finite over S , there exists $t' \in R$ such that $s := tt'$ is a non-zero element of S , and hence $sf = rt'$. Therefore, $s*a = sf*1 = rt'*1$, showing that $*a \in \text{sat}_S(*1; \mathbf{F}_{q*}R)$. The converse follows along the same lines: take $*a \in \text{sat}_S(*1; \mathbf{F}_{q*}R)$, so that $s*a = r*1$ for some non-zero $s \in S$ and $a, r \in R$. This means that $*s^qa = *r^q$, whence $s^qa = r^q$, so that $a = (r/s)^q$, and hence $f := r/s \in \mathbf{F}_q^{\text{int}}(R)$. It follows that $*a = *f^q = f*1$, as we needed to show. \square

4.4. Remark. Even if R is not a domain, it is easy to see that $\text{sat}_S(*1; \mathbf{F}_{q*}R)$ is closed under multiplication, and hence is in fact an R -subalgebra of $\mathbf{F}_{q*}R$.

Suppose again that R is a domain. Let us call, in general, regardless of characteristic, an element $f \in K$ *power-integral*, if $f^m \in K$, for all $m \gg 0$. Clearly, if R has positive characteristic, then a power-integral element is F -integral. Note, however, that power-integral elements are not closed under addition and so in general do not form a ring; instead, we have to take the R -algebra they generate, denoted R^{pow} . Clearly, R^{pow} lies in the integral closure \tilde{R} of R , and if R has positive characteristic, then

$$(7) \quad R^{\text{pow}} \subseteq \mathbf{F}^{\text{int}}(R).$$

We expect that in the toric case, (7) is an equality (see Example 5.6). For instance, in the affine case, we know that \tilde{R} is again toric. Suppose $p > [\tilde{R} : R]$, and let $f \in K$ be F-integral. Let $\Gamma(f) \subseteq \mathbb{N}$ be the semi-group of exponents m such that $f^m \in R$. Hence for some $n \gg 0$, we have $p^n \in \Gamma(f)$. Since $f \in \tilde{R}$, it satisfies a binomial integral equation of the form $T^m - u\mathbf{y}^\alpha$ with u a unit, we also get $m \in \Gamma(f)$. By assumption m and p^n are co-prime, so that $\Gamma(f)$ is co-finite, which means that f is power-integral. Note, however, that it is not even clear whether $R^{\text{pow}} = (R^{\text{pow}})^{\text{pow}}$ (it will be, if (7) is an equality).

5. HOCHSTER RINGS IN POSITIVE CHARACTERISTIC

In this section, we will provide a strategy for constructing small MCM's in characteristic p . We only will illustrate the method for the families $\mathcal{T}_{d,n,\epsilon}(\kappa)$ from the introduction, but presumably, many other cases can be treated this way (we work out one such more general case in Example 5.12). To start, we need a number-theoretic lemma:

5.1. Lemma. *Given q, m, b with $q \equiv 1 \pmod{m}$ and $b < q$, let $b(\frac{q-1}{m}) = b^{(1)}q + b^{(0)}$ be written in q -adic expansion, so that $0 \leq b^{(i)} < q$. Then $b^{(1)} + b^{(0)} < q$, and in particular, $b^{(1)}q^2 + (b^{(1)} + b^{(0)})q + b^{(0)}$ is the q -adic expansion of $b(\frac{q^2-1}{m})$.*

In fact, if we let ϵ be the 'adjusted' remainder of b modulo m , that is to say, $1 \leq \epsilon \leq m$ and $b \equiv \epsilon \pmod{m}$, then $b^{(0)} = \frac{q\epsilon-b}{m}$.

Proof. Write $q = sm + 1$ and $b = tm + \epsilon$ with $1 \leq \epsilon \leq m$. Since $b < q$, we get $t \leq s$. Hence $b^{(1)}q + b^{(0)} = b\frac{q-1}{m} = (tm + \epsilon)s = tsm + s\epsilon = t(q-1) + r = tq + s\epsilon - t$. Since $\epsilon \leq m$, we get $s\epsilon - t \leq s\epsilon \leq sm < q$. Since $t \leq s$, we get $s\epsilon - t \geq 0$ and comparing q -adic expansions, we see that $b^{(1)} = t$ and $b^{(0)} = s\epsilon - t$, and so their sum is equal to $s\epsilon < q$. The second assertion follows since $b\frac{q^2-1}{m} = (q+1)b\frac{q-1}{m}$. \square

5.2. Remark. Given $q > 1$, define the q -adic trace $\text{Tr}_q(a)$ of a number $a \in \mathbb{N}$ as the sum of its q -adic digits. The above proof shows that if $q \equiv 1 \pmod{m}$ and $b < q$ has adjusted remainder ϵ modulo m , then

$$\text{Tr}_q(b\frac{q-1}{m}) = \epsilon\frac{q-1}{m}$$

Moreover,

$$\text{Tr}_q(b\frac{q^2-1}{m}) = 2\text{Tr}_q(b\frac{q-1}{m}).$$

5.3. Theorem (The case $m = 2$). *Let $S = \kappa[[\mathbf{y}]]$ be a power series in d variables \mathbf{y} where κ is field of characteristic $p \neq 2$. Let \mathbf{u} be an n -tuple of variables, let $a_i, b_{ij} \in S$ be units and assume $a_i a_j = b_{ij}^2$ for $i < j$. Let α_i be d -tuples of (positive) exponents, such that $\alpha_i + \alpha_j = 2\beta_{ij}$, for some β_{ij} and all $i < j$. Let $I \subseteq S[[\mathbf{u}]]$ be the ideal generated by all $u_i^2 - a_i \mathbf{y}^{\alpha_i}$ and $u_i u_j - b_{ij} \mathbf{y}^{\beta_{ij}}$ for $1 \leq i < j \leq n$. Then $R := S[[\mathbf{u}]]/I$ admits a small MCM.*

Proof. Note that R is a member of the family $\mathcal{T}_{d,n,2}(\kappa)$ from the introduction. As an S -module, R is minimally generated by 1 and all u_i , but not freely, since we have the following syzygy relations

$$(8) \quad a_i \mathbf{y}^{\alpha_i} u_j = b_{ij} \mathbf{y}^{\beta_{ij}} u_i$$

for all $i < j$. In particular, R is not free as an S -module, whence is not Cohen-Macaulay. Moreover, the S -submodule N generated by all u_i is a direct summand, and $R = S \oplus N$ as S -modules (where $S \subseteq R$ in the natural way).

Let q be a power of p which is bigger than any entry in $2\beta_{ij}$. We will use the Frobenius with respect to q , but to not overload notation, we simply will assume that p itself is sufficiently big and leave the details for small primes to the reader. Since the base change $R' := R \otimes_S \mathbf{F}_* S$ is finite over R , it suffices to show that R' admits a small MCM (note that as a ring $\mathbf{F}_* S \cong S$, so that R' is also a member of $\mathcal{T}_{d,n,2}(\kappa)$). Since the a_i are p -th powers in R' , we may assume from the start that each a_i has a p -th root in S . Our goal is to show that $Q := \text{sat}_S(*1; \mathbf{F}_* R)$ is a (small) MCM, and to this end, we only need to show that $Q \cong S^2$ for then its depth as an S -module, whence as an R -module, would be d . Note that $\mathbf{F}_* R$ is (non-freely) generated as an S -module by $\{*\mathbf{y}^\delta, *u_i \mathbf{y}^\delta\}$, where δ runs over all exponent vectors in $\{0, \dots, p-1\}^d$ and where $1 \leq i \leq n$. As we will see (see, for instance, (9) below), this set is not even minimally generating.

To verify the claim, for any $\delta \in \{0, \dots, p-1\}^d$, let $\delta^{(1)}p + \delta^{(0)}$ be the p -adic expansion of $\delta \cdot (\frac{p-1}{2})$. Hence

$$u_i * 1 = *u_i^p = *u_i a_i^{\frac{p-1}{2}} \mathbf{y}^{\alpha_i^{(1)}p + \alpha_i^{(0)}} = a_i^{\frac{p-1}{2p}} \mathbf{y}^{\alpha_i^{(1)}} *u_i \mathbf{y}^{\alpha_i^{(0)}}$$

Put $e_0 := *1$ and $e_i := *u_i \mathbf{y}^{\alpha_i^{(0)}}$, so that, by definition of S -saturation, the e_i , for $i = 0, \dots, n$, generate Q . We will show that up to a unit, the e_i , for $i = 1, \dots, n$ are all the same elements, showing that Q is in fact generated as an S -module by two elements, e_0 and e_1 , which are easily seen to be linearly independent over S , as we wanted to show. To verify the latter claim, notice that our assumptions imply that all α_i have the same adjusted remainder ϵ modulo two (recall that this means that ϵ has entries either one or two, depending on the parity of the corresponding entry in the tuples α_i). By Lemma 5.1, we have

$$\alpha_i^{(0)} = \frac{p\epsilon - \alpha_i}{2}.$$

Since we chose p large enough, the quantity

$$\frac{p\epsilon - \alpha_i}{2} - \beta_{ij} = \frac{p\epsilon - \alpha_j}{2} - \alpha_i$$

is positive, so that from (8), we get

$$(9) \quad b_{ij} e_i = b_{ij} u_i \mathbf{y}^{\beta_{ij}} \mathbf{y}^{\frac{p\epsilon - \alpha_j}{2} - \alpha_i} = a_i u_j \mathbf{y}^{\frac{p\epsilon - \alpha_j}{2}} = a_i e_j$$

proving the claim. \square

5.4. Theorem (The case m odd). *Let $S = \kappa[[\mathbf{y}]]$ be a power series in d variables \mathbf{y} over a field κ of characteristic p not dividing $m \in \mathbb{N}$. Let $a, b, c \in S$ be units and assume $ab = c^m$. Let α, β be d -tuples of (positive) exponents, such that $\alpha + \beta = m\gamma$, for some γ . Let $I \subseteq S[[u, v]]$ be the ideal generated by $u^m - a\mathbf{y}^\alpha$, $v^m - b\mathbf{y}^\beta$ and $uv - c\mathbf{y}^\gamma$. Then $R := S[[u, v]]/I$ admits a small MCM.*

Proof. After taking some finite extension of κ , we may assume that the residues of a, b in κ are m -th powers, and hence, by Hensel's Lemma, there exist units $a_0, b_0 \in S$ such that $a = a_0^m$ and $b = b_0^m$. After the change of variables $u \mapsto u/a_0$ and $v \mapsto v/b_0$, we may therefore assume that they are equal to one. Let N be the maximum of the entries in α and β and choose a power q of p such that $q \equiv 1 \pmod{m}$ and $q > mN$. We will use the Frobenius with respect to q . As in the previous proof, we show that $Q := \text{sat}_S(*1; \mathbf{F}_* R)$ is a (small) MCM over R , by proving that as an

S -module, $Q \cong S^m$. As generators of \mathbf{F}_*R we take again the standard generating set $\mathbf{E} := \{u^i \mathbf{y}^\delta, v^i \mathbf{y}^\delta\}$ with $0 \leq i < m$ and $\delta \in \{0, \dots, q-1\}^d$.

Put $e_{0,0} := *1$. Since the defining equations are binomial, there exists, for each pair $(i, j) \in \mathbb{N}^2$ a generator $e_{ij} \in \mathbf{E}$ and some $s_{ij} \in S$ such that $u^i v^j e_{0,0} = s_{ij} e_{ij}$, and these e_{ij} then generate Q over S . We will show that already $e_{0,0}, e_{1,0}, \dots, e_{m-1,0}$ generate Q as an S -module. Since $1, u, \dots, u^{m-1}$ are linearly dependent over S , this shows the claim that $Q \cong S^m$. From the defining equations, it follows that $e_{m,0} = e_{0,m} = e_{1,1}$, etc. Hence the $e_{i,0}$ and $e_{0,i}$ for $i < m$ already generate Q . So remains to show that

$$(10) \quad e_{0,i} = e_{m-i,0},$$

for all $1 \leq i \leq m-1$.

To this end, let $r := \frac{q-1}{m}$ and define for any $a < q$ and any $0 < i < m$, the remainder of ria modulo q by $a_i^{(0)}$, and the adjusted remainder of ia modulo m by \tilde{a}_i (and a similar notation for tuples). It follows from Lemma 5.1 that

$$(11) \quad a_i^{(0)} = \frac{q\tilde{a}_i - ia}{m}$$

Moreover, if $a + b \equiv 0 \pmod{m}$, then $ia \equiv (m-i)b$, showing that

$$(12) \quad \tilde{a}_i = \tilde{b}_{m-i}.$$

To determine $e_{i,0}$, we must calculate

$$u^i * 1 = *u^{qi} = *u^i \mathbf{y}^{ri\alpha} = s_i * u^i \mathbf{y}^{\alpha_i^{(0)}}$$

for some $s_i \in S$, showing that $e_{i,0} = *u^i \mathbf{y}^{\alpha_i^{(0)}}$. Likewise, we have $e_{0,i} = *v^i \mathbf{y}^{\beta_i^{(0)}}$. From the identity $u^i (uv)^{m-i} = v^{m-i} u^m$, we get a relation

$$(13) \quad u^i \mathbf{y}^{(m-i)\gamma} = v^{m-i} \mathbf{y}^\alpha$$

Fix i and let $\delta := \beta_{m-i}^{(0)} - \alpha$. Since we choose q big enough, one verifies, using (11) for β , that $\delta \geq 0$. Furthermore, using (12), we get

$$\delta = \frac{p\tilde{\beta}_{m-i} - (m-i)\beta}{m} - \alpha = \frac{p\tilde{\alpha}_i - i\alpha}{m} - (m-i)\gamma = \alpha_i^{(0)} - (m-i)\gamma.$$

Therefore, we can multiply both sides of syzygy (13) with \mathbf{y}^δ , yielding

$$u^i \mathbf{y}^{\alpha_i^{(0)}} = v^{m-i} \mathbf{y}^{\beta_{m-i}^{(0)}}$$

whence proving (10). \square

5.5. Remark. The condition that p and m be co-prime (=tameness in the terminology of §5.7 below) is presumably not necessary (see Example 5.6), and one should be able to treat it in a similar way.

5.6. Example. Let me work out just one simple example in more detail. We take $m = 3$, $d = 3$, and look at the (purely) toric ring R with defining equations $u^3 = xy^2z^3$, $v^3 = x^5yz^6$, and $uv = x^2yz^3$ over a field κ of characteristic p . Note that the highest degree is $N = 6$. We have the following two syzygy relations

$$(14) \quad vxy^2z^3 = u^2x^2yz^3 \quad \text{and} \quad ux^5yz^6 = v^2x^2yz^3$$

It follows from (14) that $\mathfrak{p} := (vy - u^2x, v^2 - ux^3z^3)R$ is a minimal prime ideal of R . Put $\bar{R} := R/\mathfrak{p}$, so that \bar{R} is a three-dimensional domain. To calculate its integral closure, observe that the morphism $\varphi: \bar{R} \rightarrow \kappa[[a, b, c]]$ given by the parametrization

$$\begin{cases} x = a^3 \\ y = b^3 \\ z = c \\ u = ab^2c \\ v = a^5bc^2 \end{cases}$$

is well-defined on \bar{R} . Since source and image of φ are both three-dimensional domains, it must also be injective. Moreover, $\frac{u}{z}$ and $\frac{v}{xz^2}$ are integral over \bar{R} , and φ can be extended to the \bar{R} -algebra R' they generate by letting $\frac{u}{z} \mapsto ab^2$ and $\frac{v}{xz^2} \mapsto a^2b$. As the image of the latter morphism is the subring $\kappa[[a^3, a^2b, ab^2, b^3, c]]$, we showed that $\text{Spec}(R')$ is non-singular, equal to the product of the normal scroll of degree three with the affine line, and hence in particular, R' is normal, whence the normalization of \bar{R} . As R' is Cohen-Macaulay, it is a small MCM for R (this is true in general for complete purely toric rings by the theory of affine toric rings). We will, however, show that there is a smaller MCM given by the theorem. We consider various characteristics.

If $p = 7$, then according to the proof, we should take $q = 49$ to make it bigger than $3N = 18$, but we shall see that already $q = 7$ works. One calculates that $u*1 = *ux^2y^4z^6$ and $u^2*1 = yz*u^2x^4yz^5$, so that Q is the submodule of \mathbf{F}_*R generated by $e_0 := *1$, $e_1 := *ux^2y^4z^6$ and $e_2 := *u^2x^4yz^5$. Indeed, $v*1 = ve_0 = xze_2$ and $v^2*1 = v^2e_0 = x^3z^3e_1$ together with $ue_0 = e_1$ and $u^2e_0 = yze_2$ are the relations among these generators. In particular, as an S -module, Q is generated by the three elements e_0 , e_1 , and e_2 , and one easily checks that they are S -linearly independent, so that $Q \cong S^3$ is indeed an MCM for R .

One verifies that \mathfrak{p} annihilates Q , so that the latter is even an MCM of \bar{R} contained in $\mathbf{F}_*\bar{R}$, given by the same generators and relations over \bar{R} . In particular, $Q = \text{sat}_S(*1; \mathbf{F}_*\bar{R}) \cong \mathbf{F}_7^{\text{int}}(\bar{R})$ by Proposition 4.3. The integral elements $\frac{u}{z}$ and $\frac{v}{xz^2}$ are not F-integral, showing that R' is bigger than Q . On the other hand, the fraction $\frac{v}{xz} = \frac{u^2}{yz}$ is 7-integral and can be seen to generate $\mathbf{F}_7^{\text{int}}(\bar{R}) = \mathbf{F}^{\text{int}}(\bar{R})$. Note that $\frac{u^2}{yz}$ is actually power-integral over R , whence F-integral in any characteristic, by (7). In particular,

$$Q \cong \mathbf{F}^{\text{int}}(\bar{R}) = \bar{R}^{\text{pow}} = \bar{R}\left[\frac{u^2}{yz}\right].$$

Next take $p = 11$. According to our proof, we should work with $q = 121$ as this is equivalent to one modulo 3, but as we shall see, we can again use the Frobenius with respect to 11. Namely, let $e_0 := *1$, $e_1 := *ux^7y^3z^{10}$ and $e_2 := *u^2x^3y^6z^9$, then we have the following relations among these

$$(15) \quad ue_0 = e_2 \quad u^2e_0 = yze_1 \quad ve_0 = xze_1 \quad v^2e_0 = x^3z^3e_2$$

showing that e_0 and e_2 generate Q (freely) as an S -module.

For $p = 13$, we cannot take $q = 13$, since it will not give the necessary relations (explicitly $u*_{13}1$ and $v^2*_{13}1$ do not belong to the same cyclic S -module, whereas according to (10) they should), and so we must take $q = 169$. If we define

$e_1 := *ux^{56}y^{112}z^{168}$ and $e_2 := *u^2x^{112}y^{55}z^{167}$, then we have again the same relations (15). Although $\frac{u^2}{yz}$ generates the F-normalization of \bar{R} , it cannot be embedded as a submodule of $\mathbf{F}_{13*}\bar{R}$, but only inside $\mathbf{F}_{169*}\bar{R}$.

Although we only gave the proof for p relative prime to m (but see Remark 5.5), we can as easily check the case $p = 3$, and we may already take $q = 3$. In that case, choosing as generators for Q the elements $e_0 := *1$, $e_1 := *xy^2$ and $e_2 := *x^2y$, we get relations $ue_0 = ze_1$, $u^2e_0 = yz^2e_2$, $ve_0 = xz^2e_2$ and $ve_0 = x^3z^4e_1$, so that Q is in fact a free summand of \mathbf{F}_*S . However, something else is different: \mathfrak{p} does no longer annihilate Q and $\mathbf{F}_3^{\text{int}}(\bar{R}) = R'$, since $\frac{u}{z}$ and $\frac{v}{xz^2}$ are 3-integral.

5.7. Bi-partite toric rings. The families $\mathcal{T}_{d,n,m}$ are special cases of the following more general construction of toric local rings: let S , as above, be an unramified d -dimensional regular local ring, for which we fix a regular system of parameters $\mathbf{y} = (y_1, \dots, y_d)$, let $\Gamma \subseteq \mathbb{N}^n$ be a (finitely generated) full semi-group (meaning that its divisible (group) hull is \mathbb{Q}^n), let $\varphi: \Gamma \rightarrow \mathbb{N}^d$ be a homomorphism of semi-groups, and let χ be an S -character on Γ , by which we mean a homomorphism from Γ to the multiplicative group S^* of units in S . Since Γ is full, taking divisible hulls leads to a linear transformation $\mathbb{Q}^n \rightarrow \mathbb{Q}^d$ extending φ ; let \mathbb{A}_φ be the $n \times d$ -matrix defining this transformation (with respect to the standard bases). This matrix is (positive) integral on Γ , meaning that $\gamma\mathbb{A}_\varphi \in \mathbb{N}^d$, for all $\gamma \in \Gamma$, and conversely, any matrix over \mathbb{Q} which is integral on Γ induces a morphism of semi-groups as above.

To this data we associate a toric local ring $\mathcal{R}(S, \Gamma, \varphi, \chi)$ given as the quotient of $S[[\mathbf{u}]]$, where $\mathbf{u} = (u_1, \dots, u_n)$, modulo the toric ideal generated by all $\mathbf{u}^\gamma - \chi(\gamma)\mathbf{y}^{\varphi(\gamma)}$, for $\gamma \in \Gamma$. For instance, Example 5.6 is obtained this way by taking the semi-group generated by $(3, 0)$, $(1, 1)$, and $(0, 3)$, letting the character χ to be trivial, and letting φ to be defined by the matrix

$$\mathbb{A}_\varphi := \begin{pmatrix} 1/3 & 2/3 & 1 \\ 5/3 & 1/3 & 2 \end{pmatrix}.$$

These toric rings have the property that their defining binomial equations are given by partitioning the tuple of variables and equating a monomial in the first tuple to a unit times a monomial in the second tuple, whence the name *bi-partite*. In fact, they give a ‘toric’ Noether normalization:

5.8. Lemma. *With S, Γ, φ, χ as above, the natural map $S \rightarrow \mathcal{R}(S, \Gamma, \varphi, \chi)$ is injective and finite, i.e., a Noether normalization. Moreover, if the sum of all entries in \mathbb{A}_φ is bigger than one (the general case), then the maximal ideal \mathfrak{n} of S is a reduction of the maximal ideal of R .*

Proof. Let $d := \dim S$ and $R := \mathcal{R}(S, \Gamma, \varphi, \chi)$. By the fullness assumption, some positive multiple of each basis vector $\epsilon_i \in \mathbb{N}^n$ lies in Γ . Let $\gamma_i := a_i\epsilon_i$ be the smallest such multiple. The binomial equation corresponding to γ_i is then $u_i^{a_i} - \chi(\gamma_i)\mathbf{y}^{\varphi(\gamma_i)}$, showing that u_i is integral over S . This already shows that the map $S \rightarrow R$ is finite, and hence $\dim R \leq d$. To prove injectivity, since S is a domain, we only need to show that $\dim R \geq d$. To this end, we may make a flat base change, and so assume that S is complete with algebraically closed residue field. By Cohen’s structure theorem, it is therefore a power series over V , where V is either a field or a complete p -ring. In the second, mixed characteristic case, we can adjoin one extra variable replacing the role of the regular parameter p , and hence assume from the start that \mathbf{y} are variables in S (see the last part in the proof of Theorem 1.4 for more details).

Let us prove the inequality $d \leq \dim R$ first in the purely toric case, i.e., when $\chi = 1$ is trivial. Let a be the common denominator of the a_i , and let $\frac{\alpha_1}{a}, \dots, \frac{\alpha_d}{a}$ be the rows of the matrix \mathbb{A}_φ with $\alpha_i \in \mathbb{N}^d$. Consider the V -algebra morphism $s: S[[\mathbf{u}]] \rightarrow S$ given by $y_i \mapsto y_i^a$ and $u_i \mapsto \mathbf{y}^{\alpha_i}$. For $\gamma = (g_1, \dots, g_d) \in \Gamma$, the image of \mathbf{u}^γ under s is \mathbf{y}^β with $\beta = g_1\alpha_1 + \dots + g_d\alpha_d = a\varphi(\gamma)$, whence is equal to the image of $\mathbf{y}^{\varphi(\gamma)}$ under s . Hence s factors through R , and since s is finite, we get $\dim R \geq d$, as required.

So assume now that we have a non-trivial character χ . Let $l_i \in V^*$ be the constant term of $\chi(\gamma_i) \in S^*$, for $i = 1, \dots, n$. Since we assumed the residue field to be algebraically closed, we can find $k_i \in V$ such that $k_i^{a_i} = l_i$ (in the mixed characteristic case, we may also have to take a finite ramification of V , but this does not change the dimension). Let \mathbf{z} be a new n -tuple of variables and let S' be the localization of $S[\mathbf{z}]$ at the prime ideal generated by the y_i and $z_i - k_i$, so that S' is regular of dimension $d + n$. Let φ' be the linear transformation with $n \times (n + d)$ -matrix $(\mathbb{A}_\varphi | \mathbb{I}_n)$, where \mathbb{I}_n is the $n \times n$ -identity matrix, and let $R' := \mathcal{R}(S, \Gamma, \varphi', 1)$ (note that φ' is indeed integral on Γ). By the purely toric case, R' has dimension $d + n$. By construction, each $z_i^{a_i} - \chi(\gamma_i)$ is a non-unit, and hence the ring $\bar{R}' := R'/(z_1^{a_1} - \chi(\gamma_1), \dots, z_n^{a_n} - \chi(\gamma_n))R'$ has dimension at least d by Krull's principal ideal theorem. So remains to show that $R = \bar{R}'$. For $\gamma = (g_1, \dots, g_n) \in \Gamma$, the corresponding defining equation for R is $\mathbf{u}^\gamma - \chi(\gamma)\mathbf{y}^{\varphi(\gamma)}$, whereas for R' , it is $\mathbf{u}^\gamma - \mathbf{y}^{\varphi(\gamma)}\mathbf{z}^\gamma$, and we want to show that they are the same in \bar{R}' . Indeed, in \bar{R}' , we have

$$\begin{aligned} \mathbf{z}^\gamma &= z_1^{g_1} \dots z_n^{g_n} \\ &= (z_1^{a_1})^{\frac{g_1}{a_1}} \dots (z_n^{a_n})^{\frac{g_n}{a_n}} \\ &= \chi(\gamma_1)^{\frac{g_1}{a_1}} \dots \chi(\gamma_n)^{\frac{g_n}{a_n}} \\ &= \chi\left(\frac{g_1}{a_1}\gamma_1 + \dots + \frac{g_n}{a_n}\gamma_n\right) = \chi(\gamma). \end{aligned}$$

As for the last assertion, let \mathfrak{m} be the maximal ideal of R . The condition on \mathbb{A}_φ implies that $|\varphi(\gamma)| \geq |\gamma|$, where we write $|\cdot|$ to mean the sum of all entries of a tuple. It follows that a monomial in \mathbf{u} of degree m lies in $\mathfrak{n}^{m+1}R$, for all m bigger than some m_0 . Hence $\mathfrak{m}^m \mathfrak{n}^{m_0} R = \mathfrak{m}^{m+m_0}$. \square

5.9. Remark. Without proof, we state that the kernel of s is the unique d -dimensional prime ideal \mathfrak{p} of R , provided each $a_i > 1$ (lest we would get binomial equations that are trivially reducible). In particular, the localization $R[\frac{1}{\mathbf{y}}]$ is a domain and \mathfrak{p} is the kernel of the localization map $R \rightarrow R[\frac{1}{\mathbf{y}}]$.

5.10. Remark. By the lemma, the regular system of parameters \mathbf{y} of S is a system of parameters on R . Therefore, the parameter degree of R is at most the length $\ell(R/\mathfrak{n}R)$ of the closed fiber, and I postulate that they are actually equal (this would follow from the more general conjecture that in an arbitrary local ring (R, \mathfrak{m}) , if \mathfrak{q} is a parameter ideal which is also a reduction of \mathfrak{m} , then $\text{pardeg}(R) = \ell(R/\mathfrak{q})$).

Let p be the characteristic of the residue field of S . We call φ *tame* if \mathbb{A}_φ has linearly independent rows (whence $n \leq d$) and no entry in \mathbb{A}_φ has denominator divisible by p (this holds trivially if $p = 0$). In particular, \mathbb{A}_φ is then defined over the residue field. The rings in Theorem 5.4 are tame, but it is not clear yet whether we need this assumption (see Remark 5.5 and Example 5.12 below). Nonetheless,

we will show in Theorem 7.7 below that the mixed characteristic ‘big’ analogue of the following conjecture holds under the additional tameness condition.

5.11. Conjecture. *Let S be a regular local ring of characteristic $p > 0$, let $\Gamma \subseteq \mathbb{N}^n$ be a full semi-group, and let $\varphi: \Gamma \rightarrow \mathbb{N}^d$ and $\chi: \Gamma \rightarrow S^*$ be respectively a homomorphism and an S -character. Then the bi-partite toric ring $R := \mathcal{R}(S, \Gamma, \varphi, \chi)$ defined by this data admits a small MCM, to wit, $\text{sat}_S(*1; \mathbf{F}_{q*}R)$, for some sufficiently high power q of p .*

*Moreover, if we are in the situation described in Remark 5.9, so that R has a unique d -dimensional prime \mathfrak{p} , and if φ is tame,⁸ then \mathfrak{p} annihilates $\text{sat}_S(*1; \mathbf{F}_{q*}R)$ and*

$$(16) \quad \text{sat}_S(*1; \mathbf{F}_{q*}R) \cong \mathbf{F}^{int}(R/\mathfrak{p}) = (R/\mathfrak{p})^{pow}.$$

Can we prove Conjecture 4.2 from this conjecture, using Lemma 2.1, by showing that any toric local ring is a quasi-deformation of a bi-partite one?

5.12. Example. Let me just work out one bi-partite example for which Conjecture 5.11 holds. Let Γ be the (full) semi-group of \mathbb{N}^2 generated by $(2, 0)$, $(1, 3)$, and $(0, 6)$, and let φ be the homomorphism into \mathbb{N}^3 given by the matrix

$$\mathbb{A}_\varphi = \begin{pmatrix} 1/2 & 1 & 2 \\ 5/6 & 1 & 1/3 \end{pmatrix}.$$

It is positive integral on Γ , as it sends $(2, 0)$, $(1, 3)$, and $(0, 6)$ respectively to $(1, 2, 4)$, $(3, 4, 3)$ and $(5, 6, 2)$. For simplicity, let χ be the trivial character. In terms of equations, using (u, v) as the variables for Γ and (x, y, z) for \mathbb{N}^3 , we get the three (bi-partite) binomial equations

$$u^2 = xy^2z^4, \quad uv^3 = x^3y^4z^3, \quad v^6 = x^5y^6z^2.$$

When $p = 11$, we can take the Frobenius with $q = 11$. Let $Q := \text{sat}_S(*1; \mathbf{F}_{11*}R)$. Note that $R/(x, y, z)R$ is an Artinian ring of length 9 (and hence, by Remark 5.10, we expect that $\text{pardeg}(R) = 9$), with (monomial) basis

$$(17) \quad \{1, u, v, uv, v^2, uv^2, v^3, v^4, v^5\},$$

and it suffices to know the action of these nine elements on $*1$ to calculate Q . Define

$$\begin{aligned} e_0 &:= *1 & e_1 &:= *vxy^{10}z^7 & e_2 &:= *v^2x^2y^9z^3 \\ e_3 &:= *v^3x^3y^8z^{10} & e_4 &:= *v^4x^4y^9z^6 & e_5 &:= *v^5x^5y^6z^2 \end{aligned}$$

then the action of the basis (17) on e_0 gives respectively

$$(18) \quad \begin{array}{ccccccccc} 1 & u & v & uv & v^2 & uv^2 & v^3 & v^4 & v^5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_0 & ze_3 & e_5 & xyz^2e_2 & xye_4 & x^2y^2z^2e_1 & x^2y^2e_3 & x^3y^3ze_2 & x^4y^4ze_1, \end{array}$$

showing that Q is generated by e_0, \dots, e_5 as an S -module. Since the v^i , for $i = 0, \dots, 5$ are linearly independent, $Q \cong S^6$, whence an MCM for R . It follows from the last assertion in Lemma 5.8 and [15, Theorem 14.13], that $\text{mult}(Q) = \text{mult}(\mathbf{y}R, Q) \leq Q/\mathbf{y}Q \cong \kappa^6$, showing that Q is very small.

Let \bar{R} be the residue ring modulo a three dimensional minimal prime. It follows from the syzygy relation $ux^3y^4z^3 = v^3xy^2z^4$ that it must contain $ux^2y^2 - v^3z$, and

⁸Example 5.6 shows the necessity of this latter assumption.

without proof, we state that this latter binomial generates the minimal prime and—for reasons I do not understand yet—again kills Q , making Q into an \bar{R} -algebra, namely $\mathbf{F}_{11}^{\text{int}}(\bar{R})$. To find 11-integral elements over \bar{R} , we take a basis element from the first row in (18) and divide it by the corresponding coefficient in the second row. Modulo the relation $ux^2y^2 = v^3z$, we find four 11-integral elements, each of which is in fact power-integral. However, this does not give all F-integral elements: for instance, $\frac{v^2}{xy^2}$ is also power-integral (the fifth relation in (18) only gives the element $\frac{v^2}{xy}$). On the other hand, $\frac{v^3}{x^2y^3z}$ is integral, but not F-integral (for $p \neq 2$), showing that Q is not the normalization of \bar{R} .

When $p = q = 13$, we get the same relations (18), provided we now take as (free) generators for Q , the elements

$$\begin{aligned} e_0 &:= *1 & e_1 &:= *v^5x^{11}y^8z^7 & e_2 &:= *v^4xy^9z^3 \\ e_3 &:= *v^3x^4y^{10}z^{12} & e_4 &:= *v^2x^7y^{11}z^8 & e_5 &:= *vx^{10}y^{12}z^4 \end{aligned}$$

and again $\mathfrak{p}Q = 0$ and $Q \cong \bar{R}^{\text{pow}}$.

Tameness is not an obstruction in this case, for when $p = 3$, we can take $q = 9$, and when we let

$$\begin{aligned} e_0 &:= *1 & e_1 &:= *v^3x^8y^6z^5 & e_2 &:= *x^3z^3 \\ e_3 &:= *v^3x^2y^6z^8 & e_4 &:= *x^6z^6 & e_5 &:= *v^3x^5y^6z^2 \end{aligned}$$

then they too freely generate Q over S . Unlike the previous case, however, half of the generators belong already to the submodule \mathbf{F}_*S ; moreover, we have slightly different relations than (18) (differences are marked in boldface):

$$\begin{array}{ccccccccc} 1 & u & v & uv & v^2 & uv^2 & v^3 & v^4 & v^5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_0 & ze_3 & e_5 & xy^2z^2e_2 & xy^2e_4 & \mathbf{x}y^2z^2e_1 & x^2y^2e_3 & x^3y^4ze_2 & x^3y^4ze_1. \end{array}$$

These relations, for instance, demonstrate the F-integrality of $\frac{v^2}{xy^2}$, which we did not see for $p = 11, 13$. It is therefore not clear yet how to calculate $\mathbf{F}^{\text{int}}(R)$ in general. Nonetheless, unlike what happened in the non-tame case in Example 5.6, we still have $\mathfrak{p}Q = 0$ here.

6. TRANSFER TO EQUAL CHARACTERISTIC ZERO

Recall that we called R *strongly Hochster*, if some scalar extension of R admits a very small MCM, that is to say, a small MCM whose multiplicity is at most the equi-parameter degree of R . In equal characteristic, this is the same as the *parameter degree* of R , that is to say, the least possible length $\ell(R/I)$ where I runs over all parameter ideals. Clearly, if R is Cohen-Macaulay, whence parameter degree and multiplicity agree, a very small MCM exists, to wit, R itself, and so any local Cohen-Macaulay ring is strongly Hochster.

Proof of Theorem 1.2. Let R be a d -dimensional Noetherian local ring of equal characteristic zero with residue field κ and parameter degree ρ . Passing to the completion and then modding out a minimal prime ideal of dimension d , we may moreover assume that R is a complete domain. By [1] or [18, Theorem 7.1.6], we can

find a Lefschetz's hull $L(R)$ of R , meaning that $R \rightarrow L(R)$ is a scalar extension and $L(R)$ is the ultraproduct of d -dimensional complete Noetherian local domains R_w of positive characteristic p_w (and necessarily the p_w are unbounded; note that $L(R)$, however, will not be Noetherian). Moreover, we may choose the R_w so that they have parameter degree ρ as well. By assumption, each R_w admits a scalar extension over which a small MCM module of multiplicity at most ρ exists. Replacing each R_w with this extension and taking their ultraproduct yields a possibly bigger scalar extension of R , and so we may, from the start, assume that each R_w already admits a small MCM Q_w of multiplicity at most ρ . By [17, Prop. 3.5], we can find a d -dimensional regular subring (S_w, \mathfrak{n}_w) of R_w such that $R/\mathfrak{n}_w R_w$ has length ρ . In particular, Q_w , having depth d , is free as an S_w -module, say $Q_w \cong S_w^{N_w}$. In particular, Q_w has rank at most N_w as an R_w -module. By [15, Theorem 14.8], the multiplicity is an upper bound for N_w , showing that $N_w \leq \rho$, and so, for almost all w , they are the same, say equal to N . Therefore, the ultraproduct $Q_{\mathfrak{h}}$ of the Q_w will be a finitely generated $L(R)$ -module and by Los' Theorem, free as an $L(S) := S_{\mathfrak{h}}$ -module. Let $S_{\mathfrak{h}}$, $R_{\mathfrak{h}}$, and $Q_{\mathfrak{h}}$ be the respective cataproducts of S_w , R_w and Q_w , obtained by modding out the ideal of infinitesimals (=intersection of the powers $\mathfrak{m}_{\mathfrak{h}}^n$ of the maximal ideal). By [18, 3.2.12], the homomorphisms $S \rightarrow S_{\mathfrak{h}}$ and $R \rightarrow R_{\mathfrak{h}}$ are scalar extensions.⁹ Since $S_{\mathfrak{h}} \subseteq R_{\mathfrak{h}}$ is finite (of degree ρ), so is $S_{\mathfrak{h}} \rightarrow R_{\mathfrak{h}}$. Moreover, $Q_{\mathfrak{h}}$ is a free $S_{\mathfrak{h}}$ -module, whence an MCM over $R_{\mathfrak{h}}$. \square

6.1. *Remark.* The proof shows more generally that given a class \mathcal{T} of complete local rings closed under scalar extensions and cataproducts, if each member of \mathcal{T} of positive characteristic is strongly Hochster, then so is each member of equal characteristic zero. In particular, this applies to the family $\mathcal{T}_{d,n,e,N}$ consisting of any complete toric ring in some $\mathcal{T}_{d,n,e}(\kappa)$, with κ a field, whose defining binomial equations have degree at most N , thus proving the equal characteristic zero part of Theorem 1.5.

6.2. *Remark.* For the ultraproduct argument to go through, we could weaken the very small condition to the following: given a function $f: \mathbb{N} \rightarrow \mathbb{N}$, let us say that R is f -Hochster, if it admits a small MCM of multiplicity at most $f(\rho)$, where $\rho := \text{pardeg}(R)$ is the parameter degree of R . Let \mathcal{T} be a class of complete local rings as in Remark 6.1. If for some function f , each member of \mathcal{T} of positive characteristic is f -Hochster, then so is every member of equal characteristic zero.

6.3. *Remark.* Example 5.6 and the discussion after (7) suggest that we can postulate a more concrete MCM in equal characteristic zero as well. Namely, for a non-degenerate $R \in \mathcal{T}_{d,n,e}(\kappa)$ of characteristic p , when $p > 0$, there should be a unique d -dimensional (toric) prime ideal \mathfrak{p} (whose defining equations do not depend on p) and $\mathbf{F}^{\text{int}}(R/\mathfrak{p}) = (R/\mathfrak{p})^{\text{pow}}$ is a small MCM over R . Therefore, the same should be true in the cataproduct when $p = 0$. In other words, in equal characteristic zero, R admits a small MCM realized as the ring of power-integral elements over (the unmixed part of) R . As we saw, this would even in the affine case give a 'smaller' MCM, and so we postulate the following version of Hochster's classical theorem (in any characteristic):

⁹This might be also a good occasion to rectify some reference omissions from [18, 20]: I was not aware at that time that the construction and properties of cataproducts, including the result just quoted, already appeared under the name of *separated ultraproducts* in [13, 14, 23].

6.4. Conjecture. *If A is the coordinate ring of an affine toric variety $X \subseteq \mathbb{A}_\kappa^n$, then A^{pow} is Cohen-Macaulay.*

7. BIG MCM'S IN MIXED CHARACTERISTIC

In this section, we work over a fixed complete p -ring V (meaning that V is a complete discrete valuation ring with uniformizing parameter p), with perfect residue field κ . We will mainly be concerned with complete V -algebras R that are torsion-free (whence flat) over V . For any such V -algebra, we will write \bar{R} for R/pR , which therefore is a complete κ -algebra of dimension one less than R . We will write $\mathcal{W}(\cdot)$ for the (p -typical) Witt functor (the reader may consult [6] or the notes [16]). In particular, $\mathcal{W}(\kappa) = V$, and more generally, given a κ -algebra A , we get a V -algebra $\mathcal{W}(A)$ and an epimorphism $\mathcal{W}(A) \twoheadrightarrow A$ (which is just reduction modulo p if A is moreover perfect). The latter has a categorically defined multiplicative section $A \rightarrow \mathcal{W}(A)$, which we will simply denote by τ and call its *Teichmüller character*.

Fix a tuple of variables $\mathbf{y} = (y_1, \dots, y_n)$ and put $S_n := V[[\mathbf{y}]]$. Hence $\bar{S}_n := \kappa[[\mathbf{y}]]$. We are interested in properties of the ring $B_n := \mathcal{W}(\bar{S}_n^+)$, here \bar{S}_n^+ is the absolute integral closure of \bar{S}_n (=the integral closure of \bar{S} inside the algebraic closure $\kappa((\mathbf{y}))^{\text{alg}}$ of its field of fractions).

7.1. Proposition. *The ring B_n is a complete local domain with maximal ideal pB_n and there exists a homomorphism $S_n \rightarrow B_n$ which is faithfully flat. In particular, B_n is a big MCM algebra over S_n .*

Proof. Let us write $S := S_n$, $\bar{S} := \bar{S}_n$, etc. Since \bar{S}^+ is a perfect ring, the theory of strict p -rings as in [22, Chapter II], shows that B is local, torsion-free and complete with respect to the pB -adic topology, and $B/pB = \bar{S}^+$. Suppose $ab = 0$ in B with $a, b \in B$ non-zero. Since B is complete, whence pB -adically separated, we can write $a = p^k a'$ and $b' = p^m b'$ with $a', b' \notin pB$. Since B has no p -torsion, $a'b' = 0$. Taking residues modulo pB , we get $a'b' = 0$ in \bar{S}^+ , and since the latter is a domain, we get, say, $a' = 0$ in \bar{S}^+ , whence $a' \in pB$, contradiction. So B is a domain. Finally, let $\tilde{\mathbf{y}} := \tau(\mathbf{y})$, which therefore is a lifting of (the image of) \mathbf{y} in \bar{S}^+ . Since $\kappa \subseteq \bar{S}^+$ we get $V = \mathcal{W}(\kappa) \rightarrow \mathcal{W}(\bar{S}^+) = B$, and we can extend this homomorphism by sending \mathbf{y} to $\tilde{\mathbf{y}}$. Note that (p, \mathbf{y}) is a B -regular sequence: indeed, p is B -regular, and $B/pB = \bar{S}^+$ is known to be a big MCM \bar{S} -algebra by the work of Hochster-Huneke. Hence B is a big MCM S -algebra, and the structure morphism $S \rightarrow B$ must therefore be flat. \square

Witt transforms. Let R be a torsion-free, complete local V -algebra with residue field κ . Hence $R \cong S_n/I$ for some ideal $I \subseteq S_n$. Let $\bar{R} := R/pR$ and consider its Witt ring $\mathcal{W}(\bar{R})$. Since $\mathcal{W}(\bar{R})$ is a V -algebra, we have a canonically defined homomorphism $j: S_n \rightarrow \mathcal{W}(\bar{R})$ sending \mathbf{y} to $\tau(\mathbf{y})$. We define the *Witt transform* of R , denoted $R^{\mathcal{W}}$, as the image of S_n under j , that is to say, $R^{\mathcal{W}} \cong S_n/I^{\mathcal{W}}$, where $I^{\mathcal{W}}$ is the kernel of j (called the *Witt transform* of I). From the definitions, it is clear that

$$(19) \quad I^{\mathcal{W}} \subseteq I + pS_n.$$

It is not hard to see (or using Proposition 7.1, since $S_n \rightarrow B_n$ factors through $\mathcal{W}(\bar{S}_n)$) that $S_n^{\mathcal{W}} = S_n$. It also follows from the general theory that $\mathcal{W}(\bar{R})$, whence its subring $R^{\mathcal{W}}$, has no p -torsion, so that $R^{\mathcal{W}}$ is again a torsion-free, complete local

V -algebra with residue field κ , and so we have defined a functor $(\cdot)^\mathcal{W}$ on the latter class (the functoriality follows from that of $\mathcal{W}(\cdot)$).¹⁰

7.2. Lemma. *Let $R = S_n/I$ be torsion-free and $R^\mathcal{W}$ its Witt-transform. Then $R^\mathcal{W}$ is again torsion-free, has the same dimension as R , and $\bar{R} \cong R^\mathcal{W}/pR^\mathcal{W}$. In order for $R = R^\mathcal{W}$, it suffices that $I \subseteq I^\mathcal{W}$.*

Proof. Let $d := \dim R - 1$ and let $\tilde{\mathbf{y}} := \tau(\mathbf{y})$, so that $I^\mathcal{W}$ is the kernel of the homomorphism $S_n \rightarrow \mathcal{W}(\bar{R}): \mathbf{y} \rightarrow \tilde{\mathbf{y}}$. By Noether normalization, there exists a finite, injective homomorphism $S_d \subseteq R$, and we may choose it so that it is the restriction of $S_n \rightarrow R$ onto the first d variables (via $S_d \subseteq S_n$). The homomorphism $S_d \rightarrow R^\mathcal{W}$ is injective, for if $f \in S_d$ would be a non-zero element whose image in $R^\mathcal{W}$ is zero, then, after factoring out a power of p , we may assume that $f \notin pS_d$ but $f(\tilde{\mathbf{y}}) = 0$. Hence, the image of f is zero in \bar{R} , and this then already holds in \bar{S}_d since it is a subring. So $f \in pS_d$, contradiction. It follows that $R^\mathcal{W}$ has dimension at least $d + 1$. On the other hand, the restriction of the canonical epimorphism $\mathcal{W}(\bar{R}) \rightarrow \bar{R}$ to $R^\mathcal{W}$ must be surjective, as the composition $S_n \rightarrow \bar{S}_n \rightarrow \bar{R}$ is. It follows that the dimension of $R^\mathcal{W}$ can be at most $d + 1$, proving the first assertion.

Suppose $I \subseteq I^\mathcal{W}$ and we need to show that the other inclusion holds as well. Let $F \in I^\mathcal{W}$, so that we can write it as $F = f + pG$ with $f \in I$ and $G \in S_n$ by (19). By definition $F(\tilde{\mathbf{y}}) = 0$ and by assumption $f(\tilde{\mathbf{y}}) = 0$, whence $pG(\tilde{\mathbf{y}}) = 0$. Since $R^\mathcal{W}$ is torsion-free, this implies $G(\tilde{\mathbf{y}}) = 0$, that is to say, $G \in I^\mathcal{W}$. Hence we showed $I^\mathcal{W} = I + pI^\mathcal{W}$. By Nakayama's lemma, this implies $I = I^\mathcal{W}$. \square

7.3. Remark. It follows that if R is Cohen-Macaulay, then so is $R^\mathcal{W}$: indeed, since \bar{R} is equal to $R^\mathcal{W}/pR^\mathcal{W}$ by the lemma, and the former is Cohen-Macaulay by assumption, the claim follows as p is $R^\mathcal{W}$ -regular too.

Since $\bar{R} = R^\mathcal{W}/pR^\mathcal{W}$, we get immediately that $(R^\mathcal{W})^\mathcal{W} = R^\mathcal{W}$. This justifies calling any complete torsion-free V -algebra of the form $R^\mathcal{W}$ *Witt-closed*.

7.4. Theorem. *Any Witt-closed ring admits a big Cohen-Macaulay algebra.*

Proof. Let R be a complete, torsion-free V -algebra and let $d := \dim R - 1$. As in the previous proof, we may assume that the restriction of $S_n \rightarrow R$ onto the first d variables yields a Noether normalization $S_d \subseteq R$. This induces modulo p a finite homomorphism $\bar{S}_d \rightarrow \bar{R}$, and since $\dim \bar{R} = d$, this is again injective. Let \mathfrak{p} be a d -dimensional minimal prime of \bar{R} , then the composition $\bar{S}_d \rightarrow \bar{R}/\mathfrak{p}$ is still finite and injective. In particular, these two domains have the same absolute integral closure \bar{S}_d^+ . Therefore, the composition $\bar{R} \rightarrow \bar{R}/\mathfrak{p} \subseteq \bar{S}_d^+$ induces a homomorphism $\mathcal{W}(\bar{R}) \rightarrow B_d = \mathcal{W}(\bar{S}_d^+)$, and hence by restriction, we get a homomorphism $R^\mathcal{W} \rightarrow B_d$. Moreover, the composition $S_d \subseteq S_n \rightarrow R^\mathcal{W} \rightarrow B_d$, is the restriction of the homomorphism $S_n \rightarrow B_n$ onto the first d variables, and hence is flat by Proposition 7.1. Since $R^\mathcal{W}$ has dimension $d + 1$ by Lemma 7.2, we see that B_d is a

¹⁰Technically speaking, we defined a functor on pairs (S_n, I) such that S_n/I is torsion-free, since it is not clear at this point whether $R^\mathcal{W}$ depends on the representation $R = S_n/I$.

big Cohen-Macaulay module over it. Here is the relevant commutative diagram

$$\begin{array}{ccccc}
 S_n & \xrightarrow{j} & \mathcal{W}(\bar{S}_n) & \longrightarrow & B_n \\
 & \searrow & \uparrow & & \uparrow \\
 & & R^{\mathcal{W}} \longrightarrow \mathcal{W}(\bar{R}) & & \\
 & \nearrow & \downarrow & & \downarrow \\
 S_d & \xrightarrow{j} & \mathcal{W}(\bar{S}_d) & \longrightarrow & B_d
 \end{array}$$

□

7.5. Proof of Theorem 1.4. Let V be a complete p -ring and $S := V[[\mathbf{y}]]$. After a change of variables, we may take $\mathbf{z} := (\mathbf{y}, p)$ as system of parameters. Let us first treat the case that I is generated by sum/differences of monomials in the variables \mathbf{y} only, and put $R := V[[\mathbf{y}]]/I$. The claim follows from Theorem 7.4 once we showed that $I = I^{\mathcal{W}}$, and this will follow from the multiplicativity of the Teichmüller representatives: if $\mathbf{y}^\alpha \pm \mathbf{y}^\beta \in I$, then $\mathbf{y}^\alpha = \pm \mathbf{y}^\beta$ in \bar{R} , and hence $\tau(\mathbf{y})^\alpha = \pm \tau(\mathbf{y})^\beta$ in $\mathcal{W}(\bar{R})$. This shows that $\mathbf{y}^\alpha \pm \mathbf{y}^\beta$ lies in the kernel $I^{\mathcal{W}}$ of the homomorphism $V[[\mathbf{y}]] \rightarrow \mathcal{W}(\bar{R})$, showing that $I \subseteq I^{\mathcal{W}}$, and the result now follows from Lemma 7.2.

For the general purely toric case, I is generated by binomials of the form $\mathbf{z}^\alpha \pm \mathbf{z}^\beta$. Let $S' := S[[t]]$ and put $\mathbf{x} := (\mathbf{y}, t)$. Let I' be the ideal in S' generated by the binomials $\mathbf{x}^\alpha \pm \mathbf{x}^\beta$, for each $\mathbf{z}^\alpha \pm \mathbf{z}^\beta \in I$. By the previous case, $R' := S'/I'$ admits a big MCM. Since $t - p$ is a parameter on R' and $R = R'/(t - p)R'$, we are done by Lemma 2.1. □

7.6. Remark. The proof shows that the same result holds for any $\mu(V)$ -toric ring, where $\mu(V)$ is the ‘circle group’ consisting of all n -th roots of unity in V , since the latter are Teichmüller representatives, and even more generally, for any $\tau(\kappa)$ -toric ring.

7.7. Theorem. Let V be a complete p -ring and $S = V[[y_2, \dots, y_d]]$ with regular system of parameters (p, y_2, \dots, y_d) , let $\Gamma \subseteq \mathbb{N}^n$ with $n \leq d$ be a full semi-group, and let $\varphi: \Gamma \rightarrow \mathbb{N}^d$ and $\chi: \Gamma \rightarrow S^*$ be respectively a tame homomorphism of semi-groups and an S -character. Then the bi-partite toric ring $R := \mathcal{R}(S, \Gamma, \varphi, \chi)$ admits a big MCM.

Proof. Without loss of generality, we may assume that the residue field κ is moreover algebraically closed. As in the previous case, we may adjoin one more variable y_1 and prove the result for the (partial regular system of parameters) $\mathbf{y} := (y_1, \dots, y_d)$ (that is to say, the defining equations do not involve powers of p). Let $\bar{\chi}: \Gamma \rightarrow \kappa$ be the composition of γ with the residue map $S^* \rightarrow \kappa^*$. Since κ is algebraically closed, κ^* is divisible, and so $\bar{\chi}$ extends to a true character $\mathbb{Q}^n \rightarrow \kappa^*$, which we continue to denote by $\bar{\chi}$. Let $\epsilon_i \in \mathbb{N}^n$ be the standard basis and let $l_i := \bar{\chi}(\epsilon_i)$. By fullness, some positive multiple of each ϵ_i lies in Γ ; let $\gamma_i := a_i \epsilon_i$ be the least such positive multiple. In particular, $\bar{\chi}(\gamma_i) = l_i^{a_i}$ and the i -th row of the matrix \mathbb{A}_φ associated to φ has common denominator dividing a_i . The tameness assumption implies that no a_i is divisible by p and hence that \mathbb{A}_φ is defined over κ and has maximal rank. In particular, by Hensel’s Lemma, we can find units $g_i \in S$

such that $\chi(\gamma_i) = g_i^{a_i}$. Consider the system of equations

$$\mathbf{t}^{\varphi(\epsilon_i)} = g_i \quad \text{for } i = 1, \dots, n$$

over S in the n unknowns $\mathbf{t} := (t_1, \dots, t_d)$. Upon adding some dummy variables, we may assume that $n = d$. The residue in κ of the determinant of the Jacobian of this matrix is of the form $\mathbf{t}^\beta \det(\mathbb{A}_\varphi)$, for some $\beta \in \mathbb{N}^d$, whence is non-zero. By the multivariate Hensel's Lemma, we can therefore find a solution (z_1, \dots, z_d) of the above system in S (note that each z_i is necessarily a unit). Put $h_i := z_i^{a_i}$ and $\mathbf{h} = (h_1, \dots, h_d)$. It follows that $\mathbf{h}^{\varphi(\gamma)} = \chi(\gamma)$, for every $\gamma \in \Gamma$. Hence we can make the change of variables $y_i \mapsto h_i y_i$, reducing to the case that χ is the trivial character, that is to say, the purely toric case, so that we finish off with an application of Theorem 1.4. \square

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